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Orientation Quantity Conversion Formulas
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The 3DM-G can output orientation information in three different forms, Euler Angles, Quaternions, or a 3x3 rotation matrix (also called a coordinate transformation matrix). These are essentially equivalent except that the Euler Angles have a mathematical singularity whenever Pitch is +/-90 degrees, and are therefore unsuitable for use under conditions where such orientations are likely to occur.

This document outlines how the three different means of expressing orientation are related to one-another.

The 3DM-G fundamentally calculates orientation in the form of a rotation matrix, *M.*

$$M = \begin{bmatrix} M11 & M12 & M13 \\ M21 & M22 & M23 \\ M31 & M32 & M33 \end{bmatrix}$$

M satisfies the vector equation,

$VL = M \cdot VE$ where: *VE* is a vector expressed in the Earth-Fixed coordinate system. *VL* is the same vector expressed in the 3DM-G's local coordinate system.

When the user requests orientation in the form of Euler Angles these are derived from the rotation matrix. Euler Angles consist of the *Pitch*, *Roll* and *Yaw* angles (or equivalently, the *Elevation*, *Bank*, and *Heading*). These are calculated using the "Aircraft" or "ZYX" formulation.

$$Pitch = \theta = \arcsin(-M13)$$
$$Roll = \phi = \arctan(M23 / M33)$$
$$Yaw = \psi = \arctan(M12 / M11)$$

Note: when computing the arctan of a fraction, the possibility of quadrant ambiguity and division by zero problems occurs. Many programming languages include a function, typically called "atan2", in which the numerator and denominator of the argument are input separately. This function then correctly returns the result under all conditions. The atan2 function should be used whenever possible.

The rotation matrix corresponding to a given set of Euler angles can be calculated using:

$$M = \begin{bmatrix} \cos(\psi)\cos(\theta) & \sin(\psi)\cos(\theta) & -\sin(\theta) \\ \cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\theta)\sin(\phi) + \cos(\psi)\cos(\phi) & \cos(\theta)\sin(\phi) \\ \cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi) & \sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix}$$

where $Pitch = \theta, Roll = \phi, Yaw = \psi$

When the user requests orientation from the 3DM-G in the form of Quaternions, Q, these are derived from the rotation matrix.

$$Q = \begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix}$$ where $q0$ is the scalar component, and $q1$, $q2$, $q3$ are the vector components.

The quaternion satisfies the quaternion product equation

$VL = Q \cdot VE \cdot Q^*$ where: VE is a vector expressed in the Earth-Fixed coordinate system. VL is the same vector expressed in the 3DM-G's local coordinate system.

When converting from a rotation matrix to quaternions, there are several different formulations that can be used. In practice, the numerical resolution of these may be quite different depending on the orientation. Therefore, it is recommended that a test be made of which formulation will yield the most favorable results. This can be done in the following manner:

$$test1 = M11 + M22 + M33$$
$$test2 = M11 - M22 - M33$$
$$test3 = -M11 + M22 - M33$$
$$test4 = -M11 - M22 + M33$$

max = largest of ($test1$, $test2$, $test3$, $test4$)

if max = $test1$ then carry out:

$$S = 2\sqrt{1 + M11 + M22 + M33}$$

$$\begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix} = \begin{bmatrix} S/4 \\ (M23 - M32)/S \\ (M31 - M13)/S \\ (M12 - M21)/S \end{bmatrix}$$

if max = $test2$ then carry out:

$$S = 2\sqrt{1 + M11 - M22 - M33}$$

$$\begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix} = \begin{bmatrix} (M32 - M23)/S \\ -S/4 \\ -(M21 + M12)/S \\ -(M13 + M31)/S \end{bmatrix}$$

if max = $test3$ then carry out:

$$S = 2\sqrt{1 - M11 + M22 - M33}$$

$$\begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix} = \begin{bmatrix} (M13 - M31)/S \\ -(M21 + M12)/S \\ -S/4 \\ -(M32 + M23)/S \end{bmatrix}$$

if max = $test4$ then carry out:

$$S = 2\sqrt{1 - M11 - M22 + M33}$$

$$\begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix} = \begin{bmatrix} (M21 - M12)/S \\ -(M13 + M31)/S \\ -(M32 + M23)/S \\ -S/4 \end{bmatrix}$$

Note that $\begin{bmatrix} q0 \\ q1 \\ q2 \\ q3 \end{bmatrix} = \begin{bmatrix} -q0 \\ -q1 \\ -q2 \\ -q3 \end{bmatrix}$

It is conventional to select the signs such that $q0$ is positive.

To convert from a know quaternion to a rotation matrix, the following can be used:

$$M = 2\begin{bmatrix} q0^2 - 1/2 + q1^2 & q1q2 + q0q3 & q1q3 - q0q2 \\ q1q2 - q0q3 & q0^2 - 1/2 + q2^2 & q2q3 + q0q1 \\ q1q3 + q0q2 & q2q3 - q0q1 & q0^2 - 1/2 + q3^2 \end{bmatrix}$$

To convert between Euler Angles and Quaternions, the appropriate conversion to a rotation matrix can be made as an intermediate step.